

COCYCLES AND LOCAL PRODUCT DECOMPOSITION

BY

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Dedicated to Professor Haruo Sunouchi on his 60th birthday

ABSTRACT As an application of cocycles, we establish a relation between the classical Hardy spaces on the real line R and simply invariant subspaces on a quotient of the Bohr group. When this result is specialized suitably, it yields the well-known results concerning the elements of invariant subspaces. We also study, by using Gamelin's representation theorem, unitary functions which are the values of cocycles.

1. Preliminaries. Let K be a compact abelian group, not a circle, dual to a subgroup Γ of the discrete real line R_d . For each t in R , e_t is the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for all λ in Γ . Choose and fix a positive γ in Γ , and let K_γ be the compact subgroup consisting of all x in K such that $x(\gamma) = 1$. Then K may be identified measure-theoretically, and almost topologically, with $K_\gamma \times [0, 2\pi/\gamma]$ via the mapping $y + e_s$ to (y, s) . We suppose for simplicity that 2π lies in Γ in this section, and §§2 and 4. Thus K may be regarded as $K_{2\pi} \times [0, 1)$. Let σ and σ_1 be the normalized Haar measures on K and $K_{2\pi}$, respectively. Then we may consider $d\sigma = d\sigma_1 \times dt$ on $K_{2\pi} \times [0, 1)$.

Our objective in this note, by using this local product decomposition, is to show the fact that a certain class of analytic functions on $K_{2\pi} \times R$ has a close connection to simply invariant subspaces on K . In the next section, our characterization of simply invariant subspaces, Theorem 2.1, is obtained. In §3 we investigate the values of cocycles and answer a question of Helson. We close with some remarks in §4.

For any simply invariant subspace $\mathfrak{N}$ of $L^2(\sigma)$, we define

$$(\mathfrak{N})_+ = \bigcap_{\lambda < 0} \chi_\lambda \mathfrak{N} \quad \text{and} \quad (\mathfrak{N})_- = \text{the closure } \bigcup_{\lambda > 0} \chi_\lambda \mathfrak{N},$$

where χ_λ denotes the character on K determined by λ in Γ . Then $\mathfrak{N}$ is called to be *normalized* if $\mathfrak{N} = (\mathfrak{N})_+$. Complex-valued functions of modulus one are said to be *unitary functions*. A *cocycle* is a unitary Borel function $A(x, t)$ on $K \times R$ which satisfies the cocycle identity

$$(1.1) \quad A(x, t + u) = A(x, t)A(x + e_t, u)$$

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for all x in K and s, t, u in R . A cocycle is *trivial* (resp. a *coboundary*) if it has the form $e^{irt}p(x)\overline{p(x+e_t)}$ (resp. $p(x)\overline{p(x+e_t)}$) for some r in R and some unitary function p on K . There exists a one-to-one correspondence between normalized simply invariant subspaces and cocycles [4, Chapter 2].

We denote by $H^p(\sigma)$ and $H^p(dt)$, $0 < p \leq \infty$, the usual Hardy spaces on K and R , respectively. It is known that $H^1(dt)$ is the space of all functions in $L^1(dt)$ whose Fourier transforms vanish on the negative real line. We let $H^\infty(dt/(1+t^2)) = H^\infty(dt)$, that is, the space of all the boundary functions of bounded analytic functions in the upper half-plane. The closure of $H^\infty(dt/(1+t^2))$ in $L^p(dt/(1+t^2))$ is denoted by $H^p(dt/(1+t^2))$, $0 < p < \infty$. Recall that the class of continuous function ϕ in $H^p(dt)$ with $|\phi(t)| = O(t^{-2})$ (as $|t| \rightarrow \infty$) is dense in $H^p(dt)$, $0 < p < \infty$ (cf. [3, Chapter II, §3]). Also recall that ϕ lies in $H^p(dt/(1+t^2))$ if and only if $\phi(t)(t+i)^{-2/p}$ lies in $H^p(dt)$.

We refer the reader to [4 and 2, Chapter VII] for further details of analyticity on compact abelian groups and to [1 and 3] for results about classical Hardy spaces.

The following lemma is a minor variation of [4, Theorem 17] so the proof will be omitted.

LEMMA 1.1. *Let $\mathfrak{N}_A$ be the normalized simply invariant subspace of $L^2(\sigma)$ associated with a cocycle A . Then for any f in $L^\infty(\sigma)$, the following are equivalent:*

- (i) f lies in $\mathfrak{N}_A$;
- (ii) the function of t , $A(y, t)f(y+e_t)$, lies in $H^\infty(dt/(1+t^2))$ for σ_1 -a.a. y in $K_{2\pi}$; and
- (iii) the function of t , $\overline{A(y, t)}\overline{f(y+e_t)}$, is orthogonal to $H^1(dt)$ for σ_1 -a.a. y in $K_{2\pi}$.

We next consider certain spaces of analytic functions on $K_{2\pi} \times R$. Let $\mathfrak{H}$ be the space of all bounded Borel functions $f(y, t)$ on $K_{2\pi} \times R$ which satisfy

(1.2) the function of t , $f(y, t)$, belongs to $H^1(dt)$ for σ_1 -a.a. y in $K_{2\pi}$, and

(1.3) $\text{ess. sup}\{|f(y, t)|; (y, t) \text{ in } K_{2\pi} \times [n, n+1]\} = O(n^{-2})$.

We denote by $\mathfrak{H}^p$, $0 < p < \infty$, the closure of $\mathfrak{H} \cap L^p(d\sigma_1 \times dt)$ in $L^p(d\sigma_1 \times dt)$, where we use the ordinary metric on $L^p(d\sigma_1 \times dt)$ when $0 < p < 1$. Let $B(y, t)$ be a unitary function on $K_{2\pi} \times R$. Then for each f in $\mathfrak{H}$, we define a bounded Borel function $\Phi_B(f)$ on K by

$$(1.4) \quad \Phi_B(f)(y, s) = \sum_{n=-\infty}^{\infty} \overline{B(y - e_n, s + n)} f(y - e_n, s + n)$$

for each (y, s) in $K_{2\pi} \times [0, 1)$. Then Φ_B is a linear mapping of $\mathfrak{H}$ into $L^\infty(\sigma)$. Moreover, for any p , $0 < p \leq 1$, it can be easily seen that the restriction of Φ_B to $\mathfrak{H} \cap L^p(d\sigma_1 \times dt)$ may be extended to a bounded linear mapping of $\mathfrak{H}^p$ into $L^p(\sigma)$ (cf. [7, Lemma 1]).

2. Cocycles and the space $\mathfrak{H}$. We may now state our main result.

THEOREM 2.1. *Let $\mathfrak{N}_A$ be the simply invariant subspace of $L^2(\sigma)$ associated with a cocycle A . Then $\Phi_A(\mathfrak{H})$ is dense in $(\mathfrak{N}_A)_-$.*

PROOF. We first note that, for each $y = (y, 0)$ in $K_{2\pi} \times [0, 1]$, $y + e_t = (y, 0) + e_t = (y + e_{[t]}, t - [t])$, where $[t]$ is the largest integer not exceeding t . It follows from the cocycle identity (1.1) that

$$\begin{aligned} A(y, t) \overline{A(y - e_{n-[t]}, n + t - [t])} &= A(y, t) \overline{A(y - e_{n-[t]}, n - [t])} \overline{A(y, t)} \\ &= \overline{A(y - e_{n-[t]}, n - [t])} \end{aligned}$$

for each (y, t) in $K_{2\pi} \times R$. Hence if $f(y, t)$ lies in $\mathcal{H}$, then we obtain

$$\begin{aligned} A(y, t) \Phi_A(f)(y + e_t) &= A(y, t) \Phi_A(f)(y + e_{[t]}, t - [t]) \\ &= \sum_{n=-\infty}^{\infty} \overline{A(y - e_{n-[t]}, n - [t])} f(y - e_{n-[t]}, n + t - [t]) \end{aligned}$$

by (1.4). Let ϕ be any function in $H^1(dt)$. Then the property (1.3) assures that

$$\begin{aligned} &\int_{-\infty}^{\infty} A(y, t) \Phi_A(f)(y + e_t) \phi(t) dt \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_{n-[t]}, n - [t])} f(y - e_{n-[t]}, n + t - [t]) \phi(t) dt \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_{n-k}, n - k)} f(y - e_{n-k}, t + n - k) \phi(t) dt \\ &= \sum_{m=-\infty}^{\infty} \sum_{n-k=m} \int_k^{k+1} \overline{A(y - e_{n-k}, n - k)} f(y - e_{n-k}, t + n - k) \phi(t) dt \\ &= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int_k^{k+1} \overline{A(y - e_m, m)} f(y - e_m, t + m) \phi(t) dt \\ &= \sum_{m=-\infty}^{\infty} \overline{A(y - e_m, m)} \int_{-\infty}^{\infty} f(y - e_m, t + m) \phi(t) dt = 0 \end{aligned}$$

for σ_1 -a.a. y in $K_{2\pi}$. Therefore, by Lemma 1.1, $\Phi_A(f)$ belongs to $\mathfrak{N}_A$. Let p be a function in $\mathfrak{N}_A$ which is orthogonal to $\Phi_A(\mathcal{H})$. We set $g(y, t) = p(y + e_t)$. Then it can be seen that

$$\int_K \overline{p(x)} \Phi_A(f)(x) d\sigma(x) = \iint_{K_{2\pi} \times R} \overline{g(y, t)} \overline{A(y, t)} f(y, t) d\sigma_1(y) \times dt = 0$$

for each f in $\mathcal{H}$. This implies that the function of t , $\overline{A(y, t)} \overline{p(y + e_t)}$, lies in $H^2(dt/(1 + t^2))$ for σ_1 -a.a. y in $K_{2\pi}$. On the other hand, since the function of t , $A(y, t)p(y + e_t)$, lies in $H^2(dt/(1 + t^2))$, it must be constant. Hence $|p|$ is constant on K . From this, we may assume that p is a unitary function on K . Thus we have $A(x, t) = p(x)p(x + e_t)$ and $\mathfrak{N}_A = pH^2(\sigma)$. This completes the proof.

We collect some corollaries following from Theorem 2.1. Recall that a cocycle $A(x, t)$ is *continuous* if $A(y, t)$ is continuous on $K_{2\pi} \times R$ as a function of (y, t) [4, Chapter 5]. Let $C_0(K_{2\pi} \times R)$ denote the space of all continuous functions on $K_{2\pi} \times R$ which vanish at infinity. We notice that $\mathcal{H} \cap C_0(K_2 \times R)$ is dense in $\mathcal{H}$ as

a subspace of $L^1(d\sigma_1 \times dt)$, and that $\Phi_A(f)$ lies in $C(K)$ for any f in

$$\mathcal{H} \cap C_0(K_{2\pi} \times R).$$

These facts easily imply the following.

COROLLARY 2.2. *Let A and $\mathfrak{M}_A$ be as in Theorem 2.1. If A is continuous, then $(\mathfrak{M}_A)_- \cap C(K)$ is dense in $(\mathfrak{M}_A)_-$.*

We next give another proof of Helson's existence theorem [4, Theorem 16; 8, 10].

COROLLARY 2.3. *Let A and $\mathfrak{M}_A$ be as in Theorem 2.1. Then $\mathfrak{M}_A$ contains a unitary function.*

PROOF. Define a function $w(t)$ in $L^1(dt)$ by

$$w(t) = \begin{cases} n^{-2} & \text{on } [n, n+1), |n| \geq 1, \\ 3 \sum_{j=1}^{\infty} j^{-2} & \text{on } [0, 1). \end{cases}$$

It is easy to see that $\log w(t)$ belongs to $L^1(dt/(1+t^2))$. Hence there is a function ϕ in $H^1(dt)$ such that $|\phi(t)| = w(t)$ (cf. [3, Chapter II, Theorem 4.4]). If we set $f(y, t) = \phi(t)$, then $\Phi_A(f)$ lies in $\mathfrak{M}_A$ and $|\Phi_A(f)| \geq \sum_{j=1}^{\infty} j^{-2}$ on K . Thus it follows from Szegő's theorem that $\mathfrak{M}_A$ contains a unitary function.

COROLLARY 2.4. *Let A and $\mathfrak{M}_A$ be as in Theorem 2.1. Then there exists a function f in $\mathfrak{M}_A$ satisfying:*

- (i) *the function of t , $f(y + e_t)$, can be extended analytically to $\{z; \operatorname{Im} z > -\sqrt{3}/2\}$ for each σ_1 -a. a. y in $K_{2\pi}$;*
- (ii) *$\log |f(x)|$ belongs to $L^1(\sigma)$; and*
- (iii) *for any positive λ in Γ , f does not lie in $\chi_\lambda \mathfrak{M}_A$.*

PROOF. Let $h(y, t) = 4/(2t - 1 + \sqrt{3}i)^2$. Then $|h(y, t)| \geq 1$ on $K_{2\pi} \times [0, 1]$, and $|h(y, t)| < 1$ otherwise. So by Theorem 2.1 we may choose an integer m such that the function $g_1 = \Phi_A(h^m)$ in $\mathfrak{M}_A$ satisfies (i) and $|g_1(y, s)| > 1$ on $K_{2\pi} \times [\frac{1}{6}, \frac{5}{6}]$. It may be assumed that g_1 has property (iii). Similarly, we can construct a function g_2 in $\mathfrak{M}_A$ which satisfies (i) and $|g_2(y, s)| > 1$ on $K_{2\pi} \times \{[0, \frac{2}{6}] \cup (\frac{4}{6}, 1)\}$. It follows from Jensen's inequality and Fubini's theorem that

$$\begin{aligned} & \frac{1}{2\pi} \int_K \int_0^{2\pi} \log |g_1(x) + e^{i\theta} g_2(x)| d\sigma(x) d\theta \\ & \geq \int_K \max(\log |g_1(x)|, \log |g_2(x)|) d\sigma(x) \geq 0. \end{aligned}$$

Thus there exists a θ in $[0, 2\pi)$ for which $f = g_1 + e^{i\theta} g_2$ has the desired properties.

The above corollary is the main step in the proof of [4, Theorem 26].

We may easily choose a unitary function B on $K_{2\pi} \times R$ such that the closure $[\Phi_B(\mathcal{H})]_2$ of $\Phi_B(\mathcal{H})$ in $L^2(\sigma)$ is a doubly invariant subspace, so it is worthwhile to note a condition under which $[\Phi_B(\mathcal{H})]_2$ is simply invariant.

THEOREM 2.5. *Let B be a unitary function on $K_{2\pi} \times R$. Then $[\Phi_B(\mathcal{H})]_2$ is simply invariant if and only if there exists a cocycle $A(x, t)$ for which*

(2.1) *the function of t , $A(y, t)\overline{B(y, t)}$, belongs to $H^\infty(dt/(1+t^2))$ for σ_1 -a.a. y in $K_{2\pi}$.*

PROOF. Suppose that there exists a cocycle A with property (2.1). Then $A\overline{B}f$ lies in $\mathcal{H}$ for each f in $\mathcal{H}$. Since $\Phi_B(f) = \Phi_A(A\overline{B}f)$, it follows from Theorem 2.1 that $[\Phi_B(\mathcal{H})]_2$ is a simply invariant subspace. Conversely, suppose that $[\Phi_B(\mathcal{H})]_2$ is simply invariant, and let A be the cocycle of $([\Phi_B(\mathcal{H})]_2)_+$. Let ϕh be a function in $\mathcal{H}$ which is the product of a function ϕ in $H^1(dt)$ times a function h in $C(K_{2\pi})$. We notice that

$$A(y, t) = A(y, [t] - n)A(y + e_{[t]-n}, t - [t] + n)$$

and

$$\begin{aligned} \Phi_B(\phi h)(y + e_t) &= \Phi_B(\phi h)(y + e_{[t]}, t - [t]) \\ &= \sum_{n=-\infty}^{\infty} \overline{B(y + e_{[t]-n}, t - [t] + n)} \phi(t - [t] + n) h(y + e_{[t]-n}). \end{aligned}$$

It follows from Lemma 1.1 and an argument similar to the proof of Theorem 2.1 that

$$\begin{aligned} &\int_{-\infty}^{\infty} A(y, t) \Phi_B(\phi h)(y + e_t) \psi(t) dt \\ &= \sum_{m=-\infty}^{\infty} A(y, -m) h(y - e_m) \\ &\quad \times \int_{-\infty}^{\infty} A(y - e_m, t + m) \overline{B(y - e_m, t + m)} \phi(t + m) \psi(t) dt \\ &= 0 \end{aligned}$$

for each ψ in $H^1(dt)$. Since h is arbitrary in $C(K_{2\pi})$, we have

$$\int_{-\infty}^{\infty} A(y - e_m, t + m) \overline{B(y - e_m, t + m)} \phi(t + m) \psi(t) dt = 0$$

for each integer m . This implies that A satisfies (2.1).

3. Cocycles and unitary functions. Let $A(x, t)$ be a cocycle on K . In [5, §4], Helson has shown that if the function of x , $A_u(x) = A(x, u)$, lies in $H^2(\sigma)$, then it must be constant. This odd result grew out of a basic problem concerning spectrum of cocycles. We provide, by using Gamelin's representation theorem, some remarks on this theorem.

In this section we do not assume 2π belongs to Γ . Let K_γ be as in §1 for a positive γ in Γ , and put $u = 2\pi/\gamma$. We denote by $\mathcal{U}(K_\gamma)$ and $\mathcal{U}(K)$ the classes of all unitary functions on K_γ and K , respectively. We first recall the definition of cocycles introduced by Gamelin (see [2, Chapter VII, §11]). For any β in $\mathcal{U}(K_\gamma)$ the cocycle

$B_\beta(x, t)$ is given explicitly for positive t by

$$(3.1) \quad B_\beta(y, t) = \begin{cases} 1 & \text{on } K_\gamma \times [0, u), \\ \prod_{j=0}^{m-1} \beta(y + e_{ju}) & \text{on } K_\gamma \times [mu, (m+1)u) \end{cases}$$

for each positive integer m , and $B_\beta(y + e_s, t) = B_\beta(y, s + t)$ for s in $[0, u)$. Then B_β is trivial if and only if there is an f in $\mathcal{U}(K_\gamma)$ and some r in R for which β can be expressed in the form $\beta(y) = e^{ir} f(y) \overline{f(y + e_u)}$ for a.a. y in K_γ [4, Chapter 4, §9]. Gamelin's representation theorem [2, Chapter VII, Theorem 11.1] asserts that every cocycle A on K has the factorization $A = B_\beta C$, where β is a function in $\mathcal{U}(K_\gamma)$, and C is a coboundary.

The following theorem shows vaguely which unitary functions on K are the values of cocycles and settles a question posed by Helson [5, §1]: *Is the class of all A_1 on K different from the class of all A_2 ?*

THEOREM 3.1. *For any positive u in R , let $\gamma = 2\pi/u$, and let $\{A_u\}$ denote the class of all the values $A_u(x) = A(x, u)$ of cocycles A . Then we obtain the following properties:*

(i) *if γ belongs to Γ , then every A_u in $\{A_u\}$ has the form*

$$A_u(x) = \beta(y) q(x) \overline{q(x + e_u)}$$

for σ -a.a. $x = (y, s)$ in $K_\gamma \times [0, u)$, where β is a function in $\mathcal{U}(K_\gamma)$ and q is a function in $\mathcal{U}(K)$;

(ii) *for each positive integer m , $\{A_u\}$ contains $\{A_{mu}\}$; and*

(iii) *if γ belongs to Γ , then for any v in $(0, u)$, there exists an A_v in $\{A_u\}$ which does not lie in $\{A_u\}$.*

PROOF. (i) is a direct consequence of Gamelin's representation theorem, so it is enough to show (ii) and (iii). We notice that if $A(x, t)$ is a cocycle, then so is $A(x + x_0, t)$ for any fixed x_0 in K , and the product of two cocycles is also a cocycle. For any positive integer m , we set

$$B(x, t) = A(x, t) A(x + e_u, t) \cdots A(x + e_{(m-1)u}, t).$$

Then it follows from the cocycle identity (1.1) that $B(x, t)$ is a cocycle which satisfies $B(x, u) = A(x, mu)$. Thus we have (ii). On the other hand, by Gamelin's representation theorem, we may choose a function β in $\mathcal{U}(K_\gamma)$ for which B_β is a nontrivial cocycle. We now show that for each v in $(0, u)$, $B_\beta(x, v)$ cannot belong to $\{A_u\}$. By Definition (3.1), it can be seen that

$$(3.2) \quad B_\beta(x, v) = \begin{cases} 1 & \text{for } x = (y, s) \text{ in } K_\gamma \times [0, u - v), \\ \beta(y) & \text{for } x = (y, s) \text{ in } K_\gamma \times [u - v, u). \end{cases}$$

Suppose to the contrary that $B_\beta(x, v)$ belongs to $\{A_u\}$. Then by (i) there are functions α in $\mathcal{U}(K_\gamma)$ and p in $\mathcal{U}(K)$ such that

$$B_\beta(x, v) = \alpha(y)p(x)\overline{p(x + e_u)}$$

for σ -a.a. $x = (y, s)$ in $K_\gamma \times [0, u)$. Therefore it follows from (3.2) and Fubini's theorem that there is an s in $[0, u - v)$ such that $\alpha(y) = \overline{p((y, s))}p((y + e_u, s))$ for a.a. y in K_γ . From this fact, we can easily see that $\beta(y) = \delta(y)\overline{\delta(y + e_u)}$ for some δ in $\mathcal{U}(K_\gamma)$, so B_β must be trivial. Thus we have a contradiction, and this completes the proof.

From (ii) and (iii) of Theorem 3.1 we have

COROLLARY 3.2. *If π belongs to Γ , then $\{A_1\}$ contains strictly $\{A_2\}$.*

4. Remarks. We recall that a Borel function f on $K_{2\pi} \times R$ is *automorphic* if $f(y, s + 1) = f(y + e_1, s)$ a.a. on $K_{2\pi} \times R$, and any Borel function on $K = K_{2\pi} \times [0, 1)$ can be extended uniquely to be automorphic on $K_{2\pi} \times R$ [2, Chapter VII, §6].

(a) Let $\tilde{H}^\infty(\sigma)$ be the space of all automorphic extensions of functions in $H^\infty(\sigma)$. The following question is interesting and probably difficult:

For any cocycle A , does there exist an f in $\mathcal{H}^1$ for which $(f + \Phi_A^{-1}(\{0\})) \cdot \tilde{H}^\infty(\sigma)$ is dense in $\mathcal{H}^1$?

This is related to the old problem of whether every simply invariant subspace is generated by one of its elements. Indeed, if we could choose such a function f , then $\Phi_A(f)$ would be a single generator of $(\mathcal{M}_A)_-$.

(b) We know that the dual space of $H^p(\sigma)$, $0 < p < 1$, has dimension one [7, 9]. By the argument of [7], Theorem 2.1 provides an extension of this result:

Let $\mathcal{M}$ be a simply invariant subspace of $L^p(\sigma)$, $0 < p < 1$. Then the dual space of $\mathcal{M}$ has at most dimension one.

(c) Let $\mathcal{M}$ be a simply invariant subspace of $L^2(\sigma)$. For any f in $\mathcal{M}$, let $\tilde{f}$ denote the automorphic extension of f to $K_{2\pi} \times R$. Then we may easily verify that there exists a unitary function q on $K_{2\pi} \times R$ such that the closure $[\tilde{f}\mathcal{H}]_1$ of $\tilde{f}\mathcal{H}$ in $L^1(d\sigma_1 \times dt)$ coincides with $q\mathcal{H}^1$. Let $p(x) = p(y, s)$ be the restriction of q to $K_{2\pi} \times [0, 1)$, and set $\beta(y, s) = q(y, s + 1)\overline{q(y + e_1, s)}$. Since $\tilde{f}$ is automorphic, β defines a unitary function on $K_{2\pi}$. We denote by C_β the cocycle defined by (3.1). Then it can be seen that the cocycle $C_\beta(x, t)p(x)\overline{p(x + e_1)}$ corresponds to the simply invariant subspace generated by f (cf. [5, §3]). Similarly, let $\tilde{\mathcal{M}}$ denote the space of all automorphic extensions of functions in $\mathcal{M}$. Then it is not hard to see that $[\tilde{\mathcal{M}} \cdot \mathcal{H}]_1 = q\mathcal{H}^1$ for some unitary function q on $K_{2\pi} \times R$. Thus in the same manner, we may find the cocycle associated with $\mathcal{M}$. This provides another naive definition of cocycles (cf. [4, Chapter 2]).

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